SUPPLEMENTARY MATERIAL FOR "ON LARGE MARKET ASYMPTOTICS FOR SPATIAL PRICE COMPETITION MODELS"

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ABSTRACT. In this supplement, we provide proofs for Propositions 1 and 2 in the paper.

1. Proof of Proposition 1

We first consider the matrix A_n . Without loss of generality, we consider the (1, 1)-element of A_n , say $A_n^{(1,1)}$. Also we assume $E[z_{1i}] = 0$ to simplify the presentation. By inserting the markup formula in (4) of the main paper, we can decompose

$$\begin{aligned} A_n^{(1,1)} &= \frac{1}{\sqrt{n}} \sum_{i=1}^n z_{1i} \sum_{j \neq i} e_1(d_{ij}) p_j \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{j \neq i} z_{1i} e_1(d_{ij}) \frac{q_j}{b_{jj}^1} + \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{j \neq i} z_{1i} e_1(d_{ij}) M C_j \gamma + \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{j \neq i} z_{1i} e_1(d_{ij}) u_j \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

For T_1 , Assumptions Q (ii) and S (i) and the law of large numbers imply

$$|T_1| \le \left\{ \sqrt{n} \max_{1 \le j \le n} \frac{q_j}{b_{jj}^1} \right\} \left(\max_{1 \le i \le n} \sum_{j \ne i} |e_1(d_{ij})| \right) \frac{1}{n} \sum_{i=1}^n |z_{1i}| = o_p(1).$$

For T_2 , observe that

$$E[T_2^2] = \frac{\gamma^2}{n} \sum_{i=1}^n \sum_{i_1=1}^n \sum_{j \neq i} \sum_{j_1 \neq i_1} E[z_{1i}z_{1i_1}MC_jMC_{j_1}]e_1(d_{ij})e_1(d_{i_1j_1})$$

$$= \frac{\gamma^2}{n} \sum_{i=1}^n \sum_{j \neq i} \{E[z_{1i}MC_i]E[z_{1j}MC_j] + E[z_{1i}^2]E[MC_j^2]\}e_1(d_{ij})^2$$

$$\leq \frac{C_1}{n} \sum_{i=1}^n \sum_{j \neq i} e_1(d_{ij})^2 \leq C_1 \max_{1 \leq i \leq n} \sum_{j \neq i} e_1(d_{ij})^2 = O(1),$$

for some $C_1 > 0$, where the inequality follows the assumption that z_{1i} and MC_i have the finite fourth moments, and the last equality follows from Assumption S (ii). Thus, Chebyshev's inequality implies $T_2 = O_p(1)$. For T_3 , let $R_i = z_{1i} \sum_{j \neq i} e_1(d_{ij})u_j$ so that $T_3 = n^{-1/2} \sum_{i=1}^n R_i$. Note that $E[R_i] = 0$,

$$E[R_i^2] = E[z_{1i}^2] \sum_{j \neq i} E[u_j^2] e_1(d_{1j})^2 = E[z_{1i}^2] E[u_i^2] \sum_{j \neq i} e_1(d_{1j})^2,$$

and

$$\begin{aligned} \operatorname{Cov}(R_1, R_2) &= \sum_{j_1 \neq 1} \sum_{j_2 \neq 2} E[z_{11} z_{12} u_{j_1} u_{j_2}] e_1(d_{1j_1}) e_1(d_{2j_2}) \\ &= \sum_{j_2 \neq 2} E[z_{11} z_{12} u_{2} u_{j_2}] e_1(d_{12}) e_1(d_{2j_2}) + \sum_{j_1 \neq 1, 2} \sum_{j_2 \neq 1, 2} E[z_{11} z_{12} u_{j_1} u_{j_2}] e_1(d_{1j_1}) e_1(d_{2j_2}) \\ &= \sum_{j_2 \neq 2} E[z_{11} z_{12} u_{2} u_{j_2}] e_1(d_{12}) e_1(d_{2j_2}) + \sum_{j_1 \neq 1, 2} \sum_{j_2 \neq 1, 2, j_1} E[z_{11} z_{12} u_{j_1} u_{j_2}] e_1(d_{1j_1}) e_1(d_{2j_2}) \\ &+ \sum_{j_1 \neq 1, 2} E[z_{11} z_{12} u_{j_1}^2] e_1(d_{1j_1}) e_1(d_{1j_2}) \\ &= 0. \end{aligned}$$

Thus, we have

$$Var(T_3) = \frac{1}{n} \sum_{i=1}^n Var(R_i) = \frac{1}{n} \sum_{i=1}^n E[R_i^2] = \frac{1}{n} \sum_{i=1}^n E[z_{1i}^2] E[u_i^2] \sum_{j \neq i} e_1(d_{1j})^2$$

$$\leq C_2 \max_{1 \leq i \leq n} \sum_{j \neq i} e_1(d_{1j})^2 = O(1),$$

for some $C_2 > 0$, where the last equality follows from Assumption S (ii). Now Chebyshev's inequality implies $T_3 = O_p(1)$. Combining these results, we obtain $A_n^{(1,1)} = O_p(1)$.

We next consider the vector b_n . Without loss of generality, we consider the first element of b_n , say

$$b_n^{(1)} = \frac{1}{\sqrt{n}} \sum_{i=1}^n z_{1i} u_i + \frac{1}{\sqrt{n}} \sum_{i=1}^n z_{1i} r_i =: T_4 + T_5.$$

For T_4 , the i.i.d. and finite fourth moments assumptions guarantees

$$E[T_4^2] = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n E[z_{1i}z_{1j}u_iu_j] = \frac{1}{n} \sum_{i=1}^n E[z_{1i}^2u_i^2] = O(1).$$

Thus, T_4 is $O_p(1)$. For T_5 , note that $E[T_5^2] = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n E[z_{1i}z_{1j}r_ir_j]$ by the i.i.d. assumption, and thus we have

$$E[T_{5}^{2}] = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\ell=L_{n}+1}^{\infty} \sum_{\ell'=L_{n}+1}^{\infty} \sum_{k\neq i}^{\infty} \sum_{k'\neq j}^{\infty} E[z_{1i}z_{1j}p_{k}p_{k'}]\tilde{\alpha}_{\ell}\tilde{\alpha}_{\ell'}e_{\ell}(d_{ik})e_{\ell'}(d_{jk'})$$

$$\leq \frac{C_{3}}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\ell=L_{n}+1}^{\infty} \sum_{\ell'=L_{n}+1}^{\infty} \sum_{k\neq i}^{\infty} \sum_{k'\neq j}^{\infty} |\tilde{\alpha}_{\ell}| |\tilde{\alpha}_{\ell'}| |e_{\ell}(d_{ik})| |e_{\ell'}(d_{jk'})|$$

$$\leq C_{3}n \left(\sum_{\ell=L_{n}+1}^{\infty} |\tilde{\alpha}_{\ell}| \right)^{2} \left(\sup_{1\leq i\leq n,\ell\in\mathbb{N}} \sum_{k\neq i}^{\infty} |e_{\ell}(d_{ik})| \right)^{2}$$

$$\leq C_{4}nL_{n}^{2-2\lambda} \left(\sup_{1\leq i\leq n,\ell\in\mathbb{N}} \sum_{k\neq i}^{\infty} |e_{\ell}(d_{ik})| \right)^{2} = O(nL_{n}^{2-2\lambda}), \qquad (1)$$

for some $C_3, C_4 > 0$, where the first inequality follows from the Cauchy-Schwarz inequality and finite fourth moments assumption, the third inequality follows from $\sum_{\ell=L_n+1}^{\infty} |\tilde{\alpha}_{\ell}| \leq$ $\sum_{\ell=L_n+1}^{\infty} C_5 \ell^{-\lambda} \leq C_5 L_n^{2-2\lambda} \text{ for some } C_5 > 0 \text{ by using Assumption S (iii), and the last equality follows from Assumption S (i). Thus, Chebyshev's inequality implies <math>T_5 = O_p(\sqrt{n}L_n^{1-\lambda})$. Combining these results, we obtain $b_n^{(1)} = O_p(\max\{1, \sqrt{n}L_n^{1-\lambda}\})$.

2. Proof of Proposition 2

Let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ be the maximum and minimum eigenvalues of a matrix A, respectively. It is sufficient to show that $\Pr\{(\hat{\theta} - \theta)'(\hat{\theta} - \theta) \leq M\} \to 0$ for each M > 0. Take any M > 0. Note that

$$(\hat{\theta} - \theta)'(\hat{\theta} - \theta) = b'_n (A_n^{-1})' A_n^{-1} b_n \ge \lambda_{\min}((A_n^{-1})' A_n^{-1}) b'_n b_n = \frac{b'_n b_n}{\lambda_{\max}(A_n A'_n)},$$

where the last equality follows from

$$\lambda_{\min}((A^{-1})'A^{-1}) = \lambda_{\min}((AA')^{-1}) = \frac{1}{\lambda_{\max}(AA')},$$

for any invertible matrix A. Thus, we have

$$\Pr\{(\hat{\theta} - \theta)'(\hat{\theta} - \theta) \le M\} \le \Pr\left\{\frac{b'_n b_n}{\lambda_{\max}(A_n A'_n)} \le M\right\}$$
$$\le \Pr\left\{\frac{b'_n b_n}{\lambda_{\max}(A_n A'_n)} \le M, \ \lambda_{\max}(A_n A'_n) \le C_n\right\} + \Pr\{\lambda_{\max}(A_n A'_n) > C_n\}$$
$$\le \Pr\{b'_n b_n \le C_n M\} + o(1) \le \frac{E[b'_n b_n]}{C_n M} + o(1),$$

where the third inequality follows from the assumption $\lambda_{\max}(A_n A'_n) \leq C_n$ w.p.a.1, and the last inequality follows from Markov's inequality. By using the definition $b_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i v_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i (r_i + u_i)$, we can decompose

$$E[b'_n b_n] = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n E[z'_i z_j r_i r_j] + \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n E[z'_i z_j u_i u_j] + \frac{2}{n} \sum_{i=1}^n \sum_{j=1}^n E[z'_i z_j r_i u_j]$$

=: $T_1 + T_2 + 2T_3$.

For T_1 , similar arguments to (1) in the proof of Proposition 1 yield

$$T_{1} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} E\left[z_{i}' z_{j} \left(\sum_{\ell=L_{n}+1}^{\infty} \tilde{\alpha}_{\ell} \sum_{h\neq i} e_{\ell}(d_{ih}) p_{h}\right) \left(\sum_{\ell=L_{n}+1}^{\infty} \tilde{\alpha}_{\ell} \sum_{k\neq j} e_{\ell}(d_{jk}) p_{k}\right)\right]$$

$$\leq O(n) \left(\sum_{\ell=L_{n}+1}^{\infty} |\tilde{\alpha}_{\ell}|\right)^{2} \left(\sup_{1\leq i\leq n,\ell\in\mathbb{N}} \sum_{j\neq i} |e_{\ell}(d_{ij})|\right)^{2} = O(nL_{n}^{2-2\lambda}),$$

where the inequality follows from the Cauchy-Schwarz inequality and finite fourth moments assumption, and the second equality follows from Assumptions S (i) and (iii).

For T_2 , the i.i.d. assumption and Cauchy-Schwarz inequality imply

$$T_2 = \frac{1}{n} \sum_{i=1}^n E[z'_i z_i u_i^2] \le \sqrt{E[||z_i||^4]} \sqrt{E[u_i^4]} = O(1).$$

For T_3 , observe that

$$T_{3} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{h \neq i}^{n} E[z_{i}'z_{j}u_{j}p_{h}] \sum_{\ell=L_{n}+1}^{\infty} \tilde{\alpha}_{\ell}e_{\ell}(d_{ih})$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j \neq i}^{n} E[z_{i}]'E[z_{j}u_{j}p_{j}] \sum_{\ell=L_{n}+1}^{\infty} \tilde{\alpha}_{\ell}e_{\ell}(d_{ij})$$

$$\leq \frac{C_{1}}{n} \sum_{i=1}^{n} \sum_{j \neq i}^{\infty} \sum_{\ell=L_{n}+1}^{\infty} |\tilde{\alpha}_{\ell}||e_{\ell}(d_{ij})|$$

$$\leq C_{1} \left(\sup_{1 \leq i \leq n, \ell \in \mathbb{N}} \sum_{j \neq i}^{\infty} |e_{\ell}(d_{ij})| \right) \left(\sum_{\ell=L_{n}+1}^{\infty} |\tilde{\alpha}_{\ell}| \right) = O(L_{n}^{1-\lambda}).$$

for some $C_1 > 0$, where the second equality follows from $E[z'_j z_j u_j] = E[z_j u_j] = 0$, the first inequality follows from the Cauchy-Schwarz inequality and finite fourth moments assumption, and the last equality follows from Assumptions S (i) and (iii).

Combining these results, $E[b'_n b_n] = O(nL_n^{2-2\lambda})$, and thus

$$\Pr\{(\hat{\theta} - \theta)'(\hat{\theta} - \theta) \le M\} \le O(nL_n^{2-2\lambda}/C_n).$$

Therefore, the conclusion follows by the assumption $nL_n^{2-2\lambda}/C_n \to 0$.

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